GENERALIZED CULLEN NUMBERS

HARVEY DUBNER
Dubner Computer Systems, Inc.
6 Forest Avenue
Paramus, New Jersey 07652

A *Cullen* number is defined as $C(N) = N2^N + 1$.

These numbers play no significant role in number theory, they are not mentioned in the textbooks that I own, and I can't find the original reference to them. I can only assume that some work was done by J. Cullen early in this century when he was writing papers.

I became acquainted with Cullen numbers during my searches for large prime numbers. Sam Yates collects and disseminates a list of *Titanic* primes, which are the largest known primes with a thousand or more digits [1]. One of the major contributors to this list is Wilfrid Keller and he ahs searched for Cullen numbers which are prime up to $N = 20{,}000$ (6025 digits) [2]. The primes he found are listed in [1] and included in Table 1. There does not appear to be anything exceptional about these primes, but there certainly is an elegance associated with their symmetrical form. However, this elegance would not be affected by changing the 2 to a more general value, defined as the *base* of the Cullen number. Hence, I investigated the generalized Cullen numbers

$$C_b(N) = Nb^N + 1$$

for primality. The results are shown in Tables 1 and 2.

The search for these primes took about 600 hours on my special number theory computer designed to handle such large numbers [3].

When I examined Table 1, generalized Cullen numbers started to become more interesting. As the base b increased, the number of primes tends to decrease as would be expected. However, for prime bases greater than 3 there seemed to be almost an absence of primes. Could it be proved that for some bases no general Cullen primes was possible? If so, these numbers would become exceptionally interesting. I then investigated larger prime bases and the results are shown in Table 3.

Table 1. Generalized Cullen Numbers

$$C_b(N) = Nb^N + 1$$

B	*N*	*Test N Maximum*
2	1, 141, 4713 (1423 digits), 5795 (1749 digits), 6611 (1994 digits), 18496 (5573 digits)	2000
3	2, 8, 32, 54, 114, 414, 1400, 1850, 2848 (1363 digits), 4874 (2330 digits)	7781
4	1, 3, 7, 33, 67, 223, 663, 912, 1383, 3777 (2278 digits)	6107
5	1242	6189
6	1, 2, 91, 185, 387, 488, 747, 800	4509
7	34, 1980, (1677 digits)	4415
8	5, 17, 23, 1911 (1730 digits)	4768
9	2	4895
10	1, 3, 9, 21, 363, 2161 (2165 digits), 4839 (4843 digits)	5228
11	10	3899
12	1, 8, 247	3260
13	None	4577
14	3, 5, 6, 9, 33, 45, 243, 252, 1798 (2064 digits)	2228
15	8, 14, 44, 154, 274, 694	3124
16	1, 3, 55, 81, 223, 1227 (1481 digits), 3012 (3631 digits), 3301 (3979 digits)	3444
17	None	3195
18	1, 3, 21, 23, 842 (1060 digits), 1683 (2116 digits)	3054

Notes: (a) The number of digits for Titanic Primes (> 1000 digits) are shown in parenthesis.
(b) The data for base 2 is due to Wilfrid Keller [1].

Table 2. Miscellaneous Titanic Generalized Cullen Primes

$$C_b(N) = Nb^N + 1$$

N	*B*	*Number Of Digits*
300	2324	1013
300	2709	1033
1000	5588	3751
1154	622	3228

Table 3. Generalized Cullen Primes for Prime Bases 19-73

b	*N*	*Test N Maximum*
19	None	1851
23	None	921
29	None	2165
31	None	1907
37	36	1077
41	None	2871
43	390	1409
47	None	2793
53	None	1733
59	220	1857
61	142	1461
67	474	1473
71	None	2667
73	None	1079

In searching for large primes, the first thing I do is to test for small factors (<10,000). Typically, three out of four candidates have small factors. If this ratio gets much higher I often go looking for primes elsewhere to avoid wasting computer time. I was startled by the frequency of small factors found in generalized Cullen numbers especially for prime bases. These frequencies are tabulated in table 4. For example, base 13 shows a relative frequency of 41, which means 40 out of 41 Cullen numbers with base 13 have small factors!
I looked for reasons for the small factors. The following is easily verified $[C \text{ is short for } C_b(N)]$:

1. for N odd and b odd, C is always divisible by 2;
2. for $N = 2 + 6x$ and $b = 3y \pm 1$, C is divisible by 3;
3. for $b = 10x \pm 1$ and $N = 10x + 4$, C is divisible by 5;
4. for $b = 10x \pm 3$ and $N = 20x + 5 \pm 1$, C is divisible by 5; and
5. for $b = Kx \pm 1$, K prime, and $N = 2KZ + K - 1$, C is divisible by K.

This relationship is particularly important since it shows that if b is next to a number that contains a prime *K,* then an infinite series of *N*'s exists for which *C is* divisible by *K*. For example, when $b = 43, b = 6*7+1 = 4*11-1$, so that *C* is divisible by 7 or 11 for two series of *N*'s.

Similar but slightly more complex relationships can be found for $b = Kx \pm 2, 3, 4, \text{etc.}$

Table 4. Relative Frequency of Small Prime Factors (< 10,000) Of Generalized Cullen Numbers

Base	*Relative Frequency*	*Base*	*Relative Frequency*
2	24.5	17	43.3
3	15.8	19	38.6
4	12.9	23	46.1
5	24.1	29	49.2
6	7.2	31	50.2
7	15.8	37	44.9
8	13.0	41	49.5
9	16.2	43	58.8
10	8.4	47	41.7
11	29.4	53	34.0
12	7.1	59	39.5
13	41.0	61	76.9
14	10.3	67	59.0
15	13.8	71	42.3
16	13.7		

The next relationship is surely the most important and was completely unexpected. It emphasizes the importance of experimentation. and observation. I noticed that for several small bases, 29 was always a factor of $C_b(28)$; that is 29 divided $28b^{28}+1$. With this hint it was not too difficult to derive:

6. for $N+1$ prime, and $(N+1,b)=1$ (which means that $N+1$ and b are relatively prime), $C_b(N)$ is always divisible by $N+1$.

This is easily proved by using Fermat's theorem, $b^{P-1}-1$ is divisible by P for P prime and $(b,P)=1$, since

$$Nb^b+1=(N+1)\ b^N-b^N+1=(N+1)\ b^N-\left(b^N-1\right)$$

and both terms are divisible by $N+1$.

For most prime bases the above conditions assure that a large percentage of generalized Cullen numbers have small prime factors. For example, for $b=31$, these conditions cause 94 out of the first 100 numbers to have small factors. The first value of N for which $C_{31}(N)$ does not have a factor less than 10,000 is 298. There are only ten values of $N<705$ that do not have such small factors. The other prime bases behave in a similar manner.
Although other conditions can be derived which assure the existence of additional

small factors, it does not appear that I will be able to prove that there is some base for which no generalized Cullen prime is possible. In any case, generalized Cullen numbers are certainly interesting and deserve a place in recreational mathematics.

References:

1. S. Yates, *Known Primes with 1000 or More Digits*, privately distributed by s. Yates, Delray Beach, Florida.
2. P. Ribenboim, *The Book of Prime Number Records*, Springer-Verlag, New York, New York, p. 283, 1988.
2. H. Dubner and R. Dubner, The Development of a Powerful, Low-Cost Computer for Number Theory Applications, *Journal of Recreational Mathematics, 18.2,* pp. 81-86, 1985-86.